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Semi-classical decay of monopoles in $N = 2$ gauge theory

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ABSTRACT

It is shown how monopoles and dyons decay on curves of marginal stability in the moduli space of vacua at weak coupling in pure $N = 2$ gauge theory with arbitrary gauge group. The analysis involves a semi-classical treatment of the monopole and rests on the fact that the monopole moduli space spaces for a magnetic charge vector equal to a non-simple root enlarge discontinuously at the curves of marginal stability. This enlargement of the moduli space describes the freedom for the monopole to be separated into stable constituent monopoles. Such decays do not occur in the associated theory with $N = 4$ supersymmetry because in this case there exist bound-states at threshold.

1. Introduction

The recent progress in understanding supersymmetric gauge theories with more than one supersymmetry, promises to lead to a complete understanding of the spectrum of BPS-saturated states (or BPS states, for short). What is particularly fascinating is that the spectrum of these states can also be probed by semi-classical methods, thus yielding a way to test the validity of the exact solutions proposed.

One of the most novel phenomenon in these theories is the possibility for the decay of BPS states in certain regions of the moduli space of vacua $\mathcal{M}_{\rm vac}$ [1,2]. Generically, these decays occur on curves of co-dimension one (for $N = 2$ supersymmetry) and so $\mathcal{M}_{\rm vac}$ is divided into regions separated by the decay curves. Since these decays always occur for kinematical reasons at threshold the decay curves are known as Curves of Marginal Stability (CMS). In principle, the spectrum of BPS states in the theory can be different in each region, with the discrepancies being due to decay processes across the CMS.

In pure $N = 2$ supersymmetric SU(2) gauge theory there is a CMS which occurs only in the strong coupling region [1,3,4]. Bilal and Ferrari [5] have shown using an ingenious symmetry argument that certain dyons do actually decay on the CMS. When matter is added in the form of hypermultiplets, it has been pointed out by Seiberg and Witten [2] that a CMS will occur under certain conditions (when the mass of a hypermultiplet and the Higgs VEV are much greater than Λ) even in the region of weak coupling and hence should be describable within a semi-classical approach. In this paper we will study another example where CMS extend into the region of weak coupling, namely pure $N = 2$ gauge theory with larger gauge groups [6]. In these theories we answer the question as to how monopole and dyon decay is described within the semi-classical approximation. These decays are of the form

$$\text{dyon} \to \text{dyon} + \text{dyon}. \tag{1.1}$$

The case with hypermultiplets is somewhat different and involves decays of the form

$$\text{dyon} \to \text{dyon} + \text{quark}, \tag{1.2}$$

and has been investigated in [7].

Of central important in the semi-classical formalism for quantizing monopoles is the moduli space of the monopole solution $\mathcal{M}_{\rm mon}$ (not to be confused with the moduli space of vacua $\mathcal{M}_{\rm vac}$). In fact, the calculation of the semi-classical spectrum of dyons can be reduced to the problem of quantum mechanics on $\mathcal{M}_{\rm mon}$ [8]. Although, in theories with arbitrary gauge groups, the structure of $\mathcal{M}_{\rm mon}$ is rather complicated, quite a lot of information on the nature of these spaces is known. First of all, in theories with a real adjoint Higgs field in the Prasad-Sommerfeld limit, one can construct monopole solutions by embeddings of the SU(2) spherically symmetric 't Hooft-Polyakov monopole [9]. These

solutions are associated to particular roots of the Lie algebra of the gauge group. One can then investigate the form of the moduli space of these solutions locally by an analysis of zero modes. For technical reasons it is actually more convenient to consider the zero modes of the Dirac equation in the back-ground of the monopole solution [10]. The bosonic, or Yang-Mills, zero modes are then related to the fermionic zero modes by supersymmetry [11].

It turns out that $\mathcal{M}_{\text{mon}}$ always has a factor of the form $\mathbb{R}^3$. This simply reflects the freedom to move the centre-of-mass of the monopole solution. If the monopole is associated to a simple root[1] (with respect to a dominant Weyl chamber defined by the Higgs VEV) then there is one additional zero mode reflecting the freedom to choose the periodic U(1) "charge angle" of the monopole. In this case the monopole is "fundamental" and has a moduli space of the form

$$\mathcal{M}_{\text{mon}} = \mathbb{R} \times S^1. \tag{1.3}$$

On the contrary, if the monopole is associated to a non-simple root $\boldsymbol{\alpha}$, i.e. has a magnetic charge vector of the form

$$\boldsymbol{\alpha}^\star = \sum_{i=1}^{r} n_i \boldsymbol{\alpha}_i^\star, \tag{1.4}$$

where $\boldsymbol{\beta}^\star = \boldsymbol{\beta}/\boldsymbol{\beta}^2$, then there are additional zero modes that reflect the fact that the solution can be deformed away from spherical symmetry [10]. In this picture such a monopole is composite and consists of $(\sum_{i=1}^{r} n_i)$ fundamental monopoles. (We are assuming that the gauge group is broken to its maximal abelian subgroup by the adjoint Higgs mechanism.) Asymptotic solutions to the equations of motion (the Bogomol'nyi equation) can be constructed by simply superimposing well-separated fundamental monopole solutions. For the case when the monopole consists of a pair of distinct fundamental monopoles, i.e. $\boldsymbol{\alpha}^\star = \boldsymbol{\alpha}_a^\star + \boldsymbol{\alpha}_b^\star$, for $a \neq b$, the moduli space has the form [12]

$$\mathcal{M}_{\text{mon}} = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}_0}{\mathbb{Z}}, \tag{1.5}$$

where $\mathbb{Z}$ is a normal subgroup of the isometry group of $\mathbb{R} \times \mathcal{M}_0$. In the above the factor of $\mathbb{R}$ is associated to a particular linear combination of the $r = \text{rank}(g)$ unbroken U(1) gauge degrees-of-freedom and $\mathcal{M}_0$ is a 4 dimensional Euclidean Taub-NUT manifold. This space was first studied for the case of the monopole associated to the non-simple root of SU(3) in unpublished work by Connell [13]. This example was subsequently discussed by Gauntlett and Lowe [14] and Lee, Weinberg and Yi [15], who also showed that $\mathcal{M}_0$ admitted a unique middle-dimensional square-integrable harmonic form, thus providing evidence for exact duality in the associated $N = 4$ theory. Lee, Weinberg and Yi [12] then went on to conjecture a form for $\mathcal{M}_0$ for solutions consisting of distinct fundmental monopoles in any Lie algebra, i.e. n_i equal to 0 or 1 only (this implies that the vector $\boldsymbol{\alpha}$

[1] We shall denote the simple roots of the Lie algebra g of the gauge group as $\boldsymbol{\alpha}_i$, $i = 1, 2, \ldots, r$.

defined in (1.4) must be a root). The conjecture was proven in the case of SU(n) gauge group by Chalmers [16] and Murray [17]. The unique middle-dimensional square-integrable harmonic forms on these spaces were constructed in a general way by Gibbons [18]. Some related work is contained in [19,20]. At the moment, the structure of $\mathcal{M}_0$ in the general case, when $\boldsymbol{\alpha}$ is any root and so the n_i's can be more than 1, is not known. However, as we shall explain below, if we take as a hypothesis that the spectrum of $N = 4$ theories exhibits exact electro-magnetic duality (or Goddard-Nuyts-Olive (GNO) duality [21]), this implies that for any multi-monopole solution for which the vector $\boldsymbol{\alpha}$, defined in (1.4), is a root of the Lie algebra then the associated manifold $\mathcal{M}_0$ admits a unique square-integrable harmonic form.

Both $N = 2$ and $N = 4$ supersymmetric gauge theories are of the type discussed above, with adjoint Higgs fields. However, in these theories the Higgs fields are also vectors under an SO($N_{\mathcal{R}}$) R-symmetry, where $N_{\mathcal{R}}$ equals 2 and 6, for $N = 2$ and $N = 4$ supersymmetries, respectively. This will have important implications for the form of the monopole moduli spaces. In particular, these moduli spaces are *not* in general identical to those theories with real Higgs because the additional zero modes corresponding to the factor $\mathcal{M}_0$ are generically stablized. However, on the CMS in $\mathcal{M}_{\rm vac}$ some of the additonal zero modes survive. So $\mathcal{M}_0$, which is generically trivial, can become a non-trivial submanifold of the moduli space of the same monopole in the real Higgs theory.

The mechanism for monopole decay is now revealed: generically a monopole (or dyon) associated to any root of the Lie algebra is stable; however, on a CMS its moduli space enlarges discontinuously and the monopole may decay into stable constituents. The condition that the monopole remains stable requires the existence of a bound-state at threshold which is described in the semi-classical formalism by the existence of a certain differential form on $\mathcal{M}_0$. For an $N = 4$ theory the form must be square-integrable and harmonic, whilst in an $N = 2$ theory the form must be a square-integrable purely holomorphic harmonic form. (It turns out that $\mathcal{M}_0$ is always a hyper-Kähler manifold, see for example [11].) This means that a knowledge of the spectrum of the $N = 4$ theory has implications for the decay of dyons in the $N = 2$ theory. More specifically if we make the assumption that GNO duality of the spectrum of BPS states is exact in the $N = 4$ theory then this implies that $\mathcal{M}_0$ always admits a unique square-integrable harmonic form. The uniqueness means that the form cannot be holomorphic, since otherwise there would be an anti-holomorphic partner. So exact GNO duality in the $N = 4$ theory implies that dyons associated to non-simple roots *actually* decay on their CMS in the $N = 2$ theory (at least in pure gauge theories with no matter). Of course, the weakness in this chain of argument is the assumption that the spectrum of the $N = 4$ theory has exact GNO duality following from a lack of knowledge of the structure of $\mathcal{M}_0$. However, the results that are known in the literature are enough to prove the result for all the simply-laced Lie groups. In the other cases, we will take the assumption of exact duality in $N = 4$ theories to be a working hypothesis.

2. Embedding monopole solutions in vector Higgs models

In this section we show how to embed the SU(2) 't Hooft-Polyakov monopole solution in a theory which has $N=2$ or $N=4$ supersymmetry and an arbitrary gauge group. The case of $N=2$, where the Higgs field is complex has been discussed in [22]. In these theories the Higgs field Φ^I is adjoint valued and also carries an additional vector index $I=1,2,\ldots,N_{\mathcal{R}}$, where $N_{\mathcal{R}}$ equals 2 and 6, for $N=2$ and $N=4$ supersymmetry, respectively. The fermion fields will play no role in the construction of the monopole solutions and so we will ignore them for the rest of this section.

The first task is to establish the form of the Bogomol'nyi bound in a vector Higgs model. The energy of a configuration involving the bosonic fields is

$$U=\frac{1}{2g}\int d^3x\,\mathrm{Tr}\left(E_i^2+B_i^2+\left(D_0\Phi^I\right)^2+\left(D_i\Phi^I\right)^2+\sum_{I<J}\left[\Phi^I,\Phi^J\right]^2\right). \tag{2.1}$$

Although we have ignored the fermion fields, supersymmetry leaves its mark in the very special form of the Higgs potential in (2.1). The energy can be re-expressed in the following form

$$U=\frac{1}{2g}\int d^3x\,\mathrm{Tr}\left(E_i^2+\left(\lambda^I B_i-D_i\Phi^I\right)^2+\left(D_0\Phi^I\right)^2+2B_i\lambda^I D_i\Phi^I+\sum_{I<J}\left[\Phi^I,\Phi^J\right]^2\right). \tag{2.2}$$

where λ^I is at this stage an arbitrary constant vector of unit length.[2]

From (2.2) we derive a bound for the energy of a configuration

$$U\geq\frac{4\pi}{g}\lambda^I Q_M^I, \tag{2.3}$$

where the magnetic charge is defined to be

$$Q_M^I=\frac{1}{4\pi}\int dS_i\,\mathrm{Tr}\left(B_i\Phi^I\right), \tag{2.4}$$

with the integral defined over a large sphere at spatial infinity. The most stringent bound is achieved by choosing $\lambda^I=Q_M^I/|Q_M^I|$, giving

$$U\geq\frac{4\pi}{g}\left|Q_M^I\right|. \tag{2.5}$$

This is the analogue of the famous Bogomol'nyi bound for magnetically charged configurations in this model. A configuration which saturates the bound must satisfy the equations

$$D_0\Phi^I=0,\qquad E_i=0,\qquad \lambda^I B_i=D_i\Phi^I,\qquad [\Phi^I,\Phi^J]=0. \tag{2.6}$$

[2] We will denote the length of an SO($N_{\mathcal{R}}$) vector as $|\lambda^I|$.

where

$$\lambda^I = \frac{Q_M^I}{|Q_M^I|}. \tag{2.7}$$

We now show how the SU(2) 't Hooft-Polyakov monopole [9], in the Prasad-Sommerfeld limit [23], may be embedded in our model to produce solutions to (2.6). For convenience we shall work in a unitary gauge where the VEV of the Higgs field on the sphere at spatial infinity is a constant adjoint-valued SO($N_{\mathcal{R}}$) vector a^I. Since for the VEV $[a^I, a^J] = 0$, the components a^I can be simultaneously diagonalized by a global gauge transformation. Denoting the Cartan generators of g by the $r = \text{rank}(g)$ vector $\boldsymbol{H}$ we have

$$\lim_{|\underline{r}|\to\infty} \Phi^I(\underline{r}) = a^I = \boldsymbol{a}^I \cdot \boldsymbol{H}, \tag{2.8}$$

which defines the $N_{\mathcal{R}}$ r-dimensional vectors $\boldsymbol{a}^I$. However, this does not completely fix the gauge symmetry since it leaves the freedom to perform discrete gauge transformations in the Weyl group of G. This freedom can be fixed, for example, by demanding that $\boldsymbol{a}^1$ is in the fundamental Weyl chamber with respect to some choice of simple roots $\boldsymbol{\alpha}_i$:

$$\boldsymbol{a}^1 \cdot \boldsymbol{\alpha}_i \geq 0, \qquad i = 1, 2, \ldots, r. \tag{2.9}$$

This defines the classical moduli space of vacua W. For theories with $N = 2$ supersymmetry, the quantum moduli space of vacua will be approximately W in the region of weak coupling, that is at distances much greater than Λ, the scale of strong coupling effects, away from the subspaces where $\boldsymbol{a}^I \cdot \boldsymbol{\alpha}_i = 0$. These are precisely the subspaces where classically one would expect enhanced gauge symmetries. In the $N = 4$ theories, we would not expect strong coupling effects to appear and the quantum moduli space should equal W for any value of the coupling. In any case, in what follows we will explicitly work away from the regions $\boldsymbol{a}^I \cdot \boldsymbol{\alpha}_i = 0$ and hence avoid some of the problems associated with monopoles in theories with long-range non-abelian fields [24].

Let the fields of the SU(2) BPS monopole solution be

$$\phi(\underline{r}) = \phi_j(\underline{r};\lambda)t_j, \qquad A(\underline{r}) = A_j(\underline{r};\lambda)t_j, \qquad A_0 = 0. \tag{2.10}$$

In unitary gauge the Higgs VEV on the sphere at spatial infinity is (say)

$$\lim_{|\underline{r}|\to\infty} \phi(\underline{r}) = \lambda t_3. \tag{2.11}$$

In the above, t_j are generators of SU(2) normalized so that $[t_i, t_j] = i\epsilon_{ijk}t_k$. This solution satisfies the Bogomol'nyi equation

$$B_i = D_i\phi. \tag{2.12}$$

We now wish to embed this solution in the vector Higgs model with gauge group G. The procedure is a generalization of the embeddings of SU(2) monopoles in higher gauge

groups with a single real Higgs field discussed by Weinberg [10] following the earlier work of Bais [25]. Take an embedding of $su(2)$ in the Lie algebra g given by

$$
\begin{aligned}
t_1 &= (2\boldsymbol{\alpha}^2)^{-1/2}\,(E_{\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}) \\
t_2 &= -i(2\boldsymbol{\alpha}^2)^{-1/2}\,(E_{\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}}) \\
t_3 &= \boldsymbol{\alpha}\cdot\boldsymbol{H}/\boldsymbol{\alpha}^2,
\end{aligned}
\tag{2.13}
$$

where $\boldsymbol{\alpha}$ is some root of g. In (2.13) we have used a Cartan-Weyl basis for g.

The embedded spherically symmetric monopole solution is given by

$$
\Phi^I(\underline{r}) = \lambda^I \phi_j\left(\underline{r}; |\boldsymbol{a}^I\cdot\boldsymbol{\alpha}|\right) t_j + \xi^I, \qquad A(\underline{r}) = A_j(\underline{r};\lambda) t_j, \tag{2.14}
$$

where λ^I is defined in (2.7) and so equals

$$
\lambda^I = \frac{\boldsymbol{a}^I\cdot\boldsymbol{\alpha}}{|\boldsymbol{a}^I\cdot\boldsymbol{\alpha}|}, \tag{2.15}
$$

and ξ^I is the constant adjoint-valued vector

$$
\xi^I = \left(\boldsymbol{a}^I - \frac{\boldsymbol{a}^I\cdot\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}\boldsymbol{\alpha}\right)\cdot\boldsymbol{H}. \tag{2.16}
$$

It is a simple matter to verify (2.6) once one notices that $[t_j,\xi^I]=0$. Furthermore the solution has the correct asymptotics at spatial infinity:

$$
\lim_{|\underline{r}|\to\infty}\Phi^I(\underline{r}) = \lambda^I\left|\boldsymbol{a}^I\cdot\boldsymbol{\alpha}\right|\frac{\boldsymbol{\alpha}\cdot\boldsymbol{H}}{\boldsymbol{\alpha}^2} + \xi^I = \boldsymbol{a}^I\cdot\boldsymbol{H}. \tag{2.17}
$$

The anti-monopole solution can be embedded in a similar way.

The magnetic charge of the solution is

$$
Q_M^I = \frac{\boldsymbol{a}^I\cdot\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2} = \boldsymbol{a}^I\cdot\boldsymbol{\alpha}^\star, \tag{2.18}
$$

where the co-root $\boldsymbol{\alpha}^\star = \boldsymbol{\alpha}/\boldsymbol{\alpha}^2$ is defind to be the magnetic charge vector of the solution. The solution has a mass which saturates the Bogomol'nyi bound

$$
M = \frac{4\pi}{g}\left|\boldsymbol{a}^I\cdot\boldsymbol{\alpha}^\star\right|. \tag{2.19}
$$

Notice that the solution can be rotated with an SO($N_{\mathcal{R}}$) transformation. However this has the effect of rotating the Higgs VEV $\boldsymbol{a}^I$ so it actually relates solutions at *different* points in the moduli space of vacua W.

For a theory with an $N=2$ supersymmetry, the R-symmetry is SO(2). In this case, it will prove more convenient to rewrite the Higgs field Φ^I as a single complex-valued field $\Phi = \Phi^1 + i\Phi^2$. In this case the VEV on the sphere at spatial infinity is $\boldsymbol{a}\cdot\boldsymbol{H}$, where now $\boldsymbol{a}$ is a complex r-dimensional vector and the Higgs field of the embedded monopole solution is

$$
\Phi(\underline{r}) = \frac{\boldsymbol{\alpha}\cdot\boldsymbol{a}}{|\boldsymbol{a}\cdot\boldsymbol{\alpha}|}\phi_j\left(\underline{r}; |\boldsymbol{a}\cdot\boldsymbol{\alpha}|\right) t_j + \left(\boldsymbol{a} - \frac{\boldsymbol{a}\cdot\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}\boldsymbol{\alpha}\right)\cdot\boldsymbol{H}. \tag{2.20}
$$

3. The kinematics of monopole decay

In this section we identify the kinematically allowed regions in the classical moduli space of vacua W where monopoles can decay. The treatment is a generalization of that for the $N = 2$ theory with SU(n) gauge group appearing in [6]. Recall that a monopole associated to a root $\boldsymbol{\alpha}$ has a magnetic charge $\boldsymbol{a}^I \cdot \boldsymbol{\alpha}^\star$. In general because the state saturates the generalized Bogomol'nyi bound it is below threshold for decay into a multi-particle state of the same magnetic charge. However, on submanifolds in the moduli space of vacua the monopole can be at threshold for decay into other stable BPS states. In fact this can occur whenever there exist two roots $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ such that

$$\left|\boldsymbol{a}^I \cdot \boldsymbol{\alpha}^\star\right| = N\left|\boldsymbol{a}^I \cdot \boldsymbol{\gamma}^\star\right| + M\left|\boldsymbol{a}^I \cdot \boldsymbol{\delta}^\star\right|, \tag{3.1}$$

where conservation of magnetic charge requires

$$\boldsymbol{\alpha}^\star = N\boldsymbol{\gamma}^\star + M\boldsymbol{\delta}^\star \tag{3.2}$$

and N and M are two positive integers ≥ 1. We denote the submanifold of W on which (3.1) is satisfied as the Curve of Marginal Stability (CMS) $C_{\boldsymbol{\gamma},\boldsymbol{\delta}}$. The condition (3.1) is equivalent to

$$\boldsymbol{a}^I \cdot \boldsymbol{\gamma} = \lambda \boldsymbol{a}^I \cdot \boldsymbol{\delta}, \qquad \lambda > 0, \tag{3.3}$$

or equivalently for a complex Higgs field

$$\frac{\boldsymbol{a}\cdot\boldsymbol{\gamma}}{\boldsymbol{a}\cdot\boldsymbol{\delta}} \in \mathbb{R} > 0. \tag{3.4}$$

Notice that (3.1) involves $N_{\mathcal{R}} - 1$ real condition, so the CMS are generically submanifolds of W with co-dimension $N_{\mathcal{R}} - 1$.

We now prove an important property of $C_{\boldsymbol{\gamma},\boldsymbol{\delta}}$. The curve only has non-trivial overlap with the interior of W if $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are either both positive or both negative roots (with respect to the fundamental Weyl chamber defined by $\boldsymbol{a}^1$ or Re$(\boldsymbol{a})$ in the case of a complex Higgs field). To see this consider the map $W \to S^1$ given by

$$\cos\theta = \frac{(\boldsymbol{\gamma}\cdot\boldsymbol{a}^I)(\boldsymbol{a}^I\cdot\boldsymbol{\delta})}{|\boldsymbol{\gamma}\cdot\boldsymbol{a}^I|\,|\boldsymbol{a}^I\cdot\boldsymbol{\delta}|}. \tag{3.5}$$

The CMS (3.3) maps to $\theta = 0$. Suppose that $\boldsymbol{\gamma}$ is a positive root and $\boldsymbol{\delta}$ is a negative root, then since $\boldsymbol{a}^1 \cdot \boldsymbol{\gamma} > 0$ and $\boldsymbol{a}^1 \cdot \boldsymbol{\delta} < 0$ the image of W can never be $\theta = 0$ which proves that $C_{\boldsymbol{\gamma},\boldsymbol{\delta}}$ has no overlap with the interior of W if $\boldsymbol{\gamma}$ is a positive root and $\boldsymbol{\delta}$ is a negative root, and vice-versa. On the contrary, if $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are both positive, or both negative, roots, then the CMS (3.3) can have non-trivial overlap with the interior of W.

So the following picture emerges. For each way of writing a positive co-root $\boldsymbol{\alpha}^\star$ as the sum of two positive co-roots $N\boldsymbol{\gamma}^\star + M\boldsymbol{\delta}^\star$, for integers N and M both ≥ 1, there exists a CMS on which the $\boldsymbol{\alpha}$ monopole is at threshold for decay into $N \times \boldsymbol{\gamma}$ plus $M \times \boldsymbol{\delta}$ monopoles. Of course, on the intersections of its different CMS a monopole can decay into states corresponding to more than two distinct roots.

4. Monopole zero modes for complex Higgs

In this section we consider the structure of the moduli space of a monopole $\mathcal{M}_{\rm mon}$. Our approach will be a generalization of that of Weinberg [10] to the case of a complex Higgs field. In particular, we shall determine under what conditions the zero modes around the spherically symmetric solution discovered in [10] are stabilized in the complex Higgs theory.

The dimension of the moduli space of a monopole is equal to the number of normalizable zero modes around the monopole solution. Rather than consider directly the equations for the zero modes of the Higgs and gauge fields about a monopole solution, it is more convenient, following Weinberg [10], to consider the zero modes of the Dirac fermion field of the $N = 2$ theory in the background of a monopole. The bosonic zero modes are then related to the fermionic zero modes by the supersymmetry left unbroken by the monopole background in such a way that for each Dirac zero mode there are two bosonic, or Yang-Mills zero modes [11].

The Dirac equation in the background of the monopole solution (2.20) is

$$\left[i\gamma^\mu D_\mu - \mathrm{Re}(\Phi) + i\gamma^5 \mathrm{Im}(\Phi)\right]\Psi = 0. \tag{4.1}$$

The monopole solution in the gauge with $A_0 = 0$ is time-independent and so we look for solutions of (4.1) of the form $\Psi(\underline{r}, t) = e^{iEt}\Psi(\underline{r})$. Using a representation

$$\gamma^j = \begin{pmatrix} -i\sigma_j & 0 \\ 0 & i\sigma_j \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.2}$$

we can write the equation for the modes as

$$H\Psi = \begin{pmatrix} \mathrm{Im}(\Phi) & i\sigma_j D_j + i\mathrm{Re}(\Phi) \\ i\sigma_j D_j - i\mathrm{Re}(\Phi) & -\mathrm{Im}(\Phi) \end{pmatrix}\Psi = E\Psi. \tag{4.3}$$

The SO(2) R-symmetry acts on the fields in the following way:

$$\Phi \mapsto e^{i\epsilon}\Phi, \qquad \Psi \mapsto e^{i\epsilon\gamma^5/2}\Psi, \qquad A_i \mapsto A_i. \tag{4.4}$$

Under such a transformation the operator H is *not* invariant because the Higgs field of the background monopole solution is not invariant. Such transformations, therefore, do not lead to a degeneracy, however, we can use them to rotate the VEV $\boldsymbol{a} \to \boldsymbol{a}'$ so that $\boldsymbol{a}' \cdot \boldsymbol{\alpha} \in \mathbb{R} > 0$. We will denote the transformed real and imaginary part of the Higgs field as P and S, respectively. The transformation is achieved by choosing the parameter $e^{i\epsilon}$ in (4.4) to be $(\boldsymbol{a}^* \cdot \boldsymbol{\alpha})/|\boldsymbol{a} \cdot \boldsymbol{\alpha}|$ and is explicitly

$$\Phi \mapsto \Phi' = (P + iS) = \frac{\boldsymbol{a}^* \cdot \boldsymbol{\alpha}}{|\boldsymbol{a} \cdot \boldsymbol{\alpha}|}\Phi. \tag{4.5}$$

We will denote the transformed Higgs VEV as

$$\boldsymbol{a}' = \frac{\boldsymbol{a}^* \cdot \boldsymbol{\alpha}}{|\boldsymbol{a} \cdot \boldsymbol{\alpha}|}\boldsymbol{a}. \tag{4.6}$$

The reason why this is useful is because the transformed Dirac equation is now related in a simple way to that of a real Higgs theory since after the transformation, the real and imaginary parts of the Higgs field are

$$\begin{aligned} P &= \phi_j(\underline{r}; |\boldsymbol{a} \cdot \boldsymbol{\alpha}|)t_j + \left(\mathrm{Re}(\boldsymbol{a}') - \frac{\boldsymbol{a}' \cdot \boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}\boldsymbol{\alpha}\right) \cdot \boldsymbol{H} \\ S &= \mathrm{Im}(\boldsymbol{a}') \cdot \boldsymbol{H}. \end{aligned} \tag{4.7}$$

So, bearing in mind that $\boldsymbol{a}' \cdot \boldsymbol{\alpha} = |\boldsymbol{a} \cdot \boldsymbol{\alpha}|$, the real part of the Higgs field is identical to the Higgs field for a monopole in a real Higgs theory with VEV $\mathrm{Re}(\boldsymbol{a}')$ and the imaginary part of the Higgs field is a constant. In the transformed variables the Bogomol'nyi equations (2.6), for time independent fields in the $A_0 = 0$ gauge, become

$$B_i = D_i P, \qquad D_i S = 0, \qquad [P, S] = 0. \tag{4.8}$$

The transformed Dirac equation is

$$H'\Psi = \begin{pmatrix} S & -\mathcal{D}^* \\ -\mathcal{D} & -S \end{pmatrix} \Psi = E\Psi, \tag{4.9}$$

where

$$\mathcal{D} = -i\sigma_j D_j + iP, \qquad \mathcal{D}^* = -i\sigma_j D_j - iP, \tag{4.10}$$

and P and S act by adjoint action on Ψ. We are now interested in the zero energy solutions of (4.9). Fortunately, we can draw on the results of Weinberg, who calculated the number of zero modes of the two operators $\mathcal{D}$ and $\mathcal{D}^*$. First of all, let us write $H' = H_0 + K$, where

$$H_0 = \begin{pmatrix} 0 & -\mathcal{D}^* \\ -\mathcal{D} & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}. \tag{4.11}$$

It follows from the Bogomol'nyi equations (4.8) that

$$\begin{aligned} \mathcal{D}^*\mathcal{D} &= -D_i^2 + P^2 - 2i\sigma_i B_i \\ \mathcal{D}\mathcal{D}^* &= -D_i^2 + P^2. \end{aligned} \tag{4.12}$$

So $\mathcal{D}\mathcal{D}^*$ is positive and has no non-trivial zero modes, implying that $\mathcal{D}^*$ itself has no such modes. On the contrary $\mathcal{D}^*\mathcal{D}$ can have zero modes given by the zero modes of $\mathcal{D}$ itself. The number of such zero modes has been calculated by Weinberg from the Callias index theorem [10]. We review the results later in this section.

If ψ is a zero mode of $\mathcal{D}$ then it follows that

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix}, \tag{4.13}$$

is a zero mode of H_0. Using $\{H_0, K\} = 0$, it follows that

$$H'^2 = H_0^2 + K^2 = \begin{pmatrix} \mathcal{D}^*\mathcal{D} + S^2 & 0 \\ 0 & \mathcal{D}\mathcal{D}^* + S^2 \end{pmatrix}, \tag{4.14}$$

and since K^2 is a positive operator the modes (4.13) are lifted if

$$S\psi = [(\mathrm{Im}(\boldsymbol{a}') \cdot H), \psi] \neq 0. \tag{4.15}$$

To conclude, the zero modes of H', and hence H, are given by the subspace of the zero modes of $\mathcal{D}$ with zero eigenvalue with respect to S.

In order to find the eigenvalues of the modes under S we need a more explicit description of the modes. Fortunately, the results we need have been established by Weinberg [10]. First of all, the real part of the VEV $\mathrm{Re}(\boldsymbol{a}')$ defines a set of simple roots $\boldsymbol{\alpha}_i'$ under which any positive root has a positive inner product with $\mathrm{Re}(\boldsymbol{a}')$. Notice that these simple roots are *not* in general the same as $\boldsymbol{\alpha}_i$ defined in section 2 with respect to $\boldsymbol{a}^1 \equiv \mathrm{Re}(\boldsymbol{a})$. To determine the number of zero modes of the monopole constructed via an $su(2)$ embedding (2.13), one expands the co-root $\boldsymbol{\alpha}^\star$ in terms of the simple co-roots $\boldsymbol{\alpha}_i'^\star = \boldsymbol{\alpha}_i'/\boldsymbol{\alpha}_i'^2$:

$$\boldsymbol{\alpha}^\star = \sum_{i=1}^{r} n_i \boldsymbol{\alpha}_i'^\star, \tag{4.16}$$

where the $n_i \in \mathbb{Z} \geq 0$. The overall number of Dirac zero modes is then $(2\sum_{i=1}^r n_i)$ and hence the number of bosonic, or Yang-Mills, zero modes is

$$4\sum_{i=1}^{r} n_i. \tag{4.17}$$

The result has a simple interpretation: in general a monopole is a composite object and the classical solution may be deformed into an asymptotic region consisting of well separated fundamental monopoles associated to the simple roots $\boldsymbol{\alpha}_i'$. For each fundamental monopole there are four bosonic zero modes corresponding to the centre-of-mass and overall U(1) charge rotation.

Following Weinberg [10], we now expand the adjoint valued field Ψ in terms of the generators of g in a Cartan Weyl basis. The zero modes can be associated to multiplets of generators under adjoint action by the $su(2)$ subalgebra of g defined by the root $\boldsymbol{\alpha}$ in (2.13). The isospin of a generator $E_{\boldsymbol{\beta}}$ is given by

$$t_3 = \frac{\boldsymbol{\alpha} \cdot \boldsymbol{\beta}}{\boldsymbol{\alpha}^2}, \tag{4.18}$$

and a multiplet is labelled by a total isospin t along with

$$y = \frac{\mathrm{Re}(\boldsymbol{a}') \cdot \boldsymbol{\beta}}{\mathrm{Re}(\boldsymbol{a}') \cdot \boldsymbol{\alpha}} - t_3, \tag{4.19}$$

which has the same value for any generator in the multiplet. The only possibilities for the total isospin are $t = 0, \frac{1}{2}, 1, \frac{3}{2}$. For each multiplet, the number of (normalizable) Dirac zero modes depends upon t and y in the following way [10]:

$$\begin{aligned}
t = \tfrac{1}{2}: \qquad & 0 < |y| < \tfrac{1}{2}, \quad \text{one} \\
& \tfrac{1}{2} < |y|, \quad \text{none} \\
t = 1: \qquad & 0 \leq |y| < 1, \quad \text{two} \\
& 1 < |y|, \quad \text{none} \\
t = \tfrac{3}{2}: \qquad & 0 < |y| < \tfrac{1}{2}, \quad \text{four} \\
& \tfrac{1}{2} < |y| < \tfrac{3}{2}, \quad \text{three} \\
& \tfrac{3}{2} < |y|, \quad \text{none.}
\end{aligned} \tag{4.20}$$

We can physically identify each multiplet in the following way. Firstly, there is always a $t = 1$ multiplet with $y = 0$ corresponding to the three generators of the $su(2)$ defining the embedding (2.13). This gives rise to two Dirac, i.e. four bosonic zeros modes, which reflect the three translational and one U(1) charge rotational degrees-of-freedom of the spherically symmetric $\boldsymbol{\alpha}$ monopole that are always present.

The other multiplets come in hermitian conjugate pairs with the opposite value of y; hence, the number of bosonic zero modes is always divisible by 4. Consider a particular pair of conjugate multiplets. We can label this pair of multiplets by $\boldsymbol{\beta}$ the root whose generator has $t_3 = t$ in the multiplet with $y > 0$. The number of bosonic zero modes associated to the pair of multiplets is equal to $4K$ for some integer $K \geq 0$. We will verify below on a case-by-case basis for each pair of multiplets $(t, \pm y)$ that the co-root $\boldsymbol{\alpha}^\star$ can always be written

$$\boldsymbol{\alpha}^\star = N\boldsymbol{\gamma}^\star + M\boldsymbol{\delta}^\star, \tag{4.21}$$

where $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are two positive roots with respect to $\boldsymbol{\alpha}'_i$ and N and M are two positive integers with $N + M = K + 1$. The interpretation of these $4K$ zero modes is now apparent, they correspond to the freedom to deform the spherically symmetric $\boldsymbol{\alpha}$ monopole into an asymptotic region which consists of $N \times \boldsymbol{\gamma}$ plus $M \times \boldsymbol{\delta}$ monopoles. Such a configuration requires $4(N + M - 1) = 4K$ relative degrees-of-freedom corresponding to relative positions and U(1) charge angles of the $N + M$ consistent monopoles. These degrees-of-freedom are manifested in the $4K$ zero modes coming from the pair of multiplets associated to $\boldsymbol{\beta}$.

We now establish (4.21) on a case-by-case basis. In the following, $\boldsymbol{\alpha}$ is always the root defining the $su(2)$ embedding and $\boldsymbol{\beta}$ is the root labelling the generator with $t_3 = t$ and $y > 0$ of a pair of $(t, \pm y)$ multiplets. We need not consider the $(t = 1, y = 0)$ multiplet associated with the centre-of-mass and overall U(1) charge rotation that is always present.

(i) $t = \frac{1}{2}$, which requires $\boldsymbol{\alpha}$ to be a long root. In this case the pair of multiplets consists of generators

$$\{E_{\boldsymbol{\beta}}, E_{\boldsymbol{\beta}-\boldsymbol{\alpha}}\} \qquad \text{and} \qquad \{E_{\boldsymbol{\alpha}-\boldsymbol{\beta}}, E_{-\boldsymbol{\beta}}\}\,. \tag{4.22}$$

There are two separate cases to consider depending on whether $\boldsymbol{\beta}$ is a long or a short root.

If $\boldsymbol{\beta}$ is a long root (which covers all the simply-laced cases) then

$$\boldsymbol{\alpha} = \boldsymbol{\beta} + (\boldsymbol{\alpha} - \boldsymbol{\beta}) \qquad \Rightarrow \qquad \boldsymbol{\alpha}^\star = \boldsymbol{\beta}^\star + (\boldsymbol{\alpha} - \boldsymbol{\beta})^\star. \tag{4.23}$$

On the other hand, if $\boldsymbol{\beta}$ is a short root, which can only occur in a non-simply-laced algebra, then

$$\boldsymbol{\alpha} = 2(\boldsymbol{\alpha} - \boldsymbol{\beta}) + (2\boldsymbol{\beta} - \boldsymbol{\alpha}) \qquad \Rightarrow \qquad \boldsymbol{\alpha}^\star = (\boldsymbol{\alpha} - \boldsymbol{\beta})^\star + (2\boldsymbol{\beta} - \boldsymbol{\alpha})^\star. \tag{4.24}$$

In both cases, the condition that $0 < |y| < \frac{1}{2}$ is equivalent to the requirement that

$$\mathrm{Re}(\boldsymbol{a}') \cdot (\boldsymbol{\alpha} - \boldsymbol{\beta}) > 0 \qquad \text{and} \qquad \mathrm{Re}(\boldsymbol{a}') \cdot (2\boldsymbol{\beta} - \boldsymbol{\alpha}) > 0, \tag{4.25}$$

and therefore implies that $\boldsymbol{\beta}$, $\boldsymbol{\alpha}-\boldsymbol{\beta}$ and $2\boldsymbol{\beta}-\boldsymbol{\alpha}$ are all positive roots with respect to $\mathrm{Re}(\boldsymbol{a}')$, as required. The number of bosonic zero modes in both cases is 4 which is consistent with a decay into 2 stable monopoles.

(ii) $t = 1$, which requires $\boldsymbol{\alpha}$ to be a short root. In this case the pair of multiplets consists of generators

$$\{E_{\boldsymbol{\beta}}, E_{\boldsymbol{\beta}-\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}-2\boldsymbol{\alpha}}\} \qquad \text{and} \qquad \{E_{2\boldsymbol{\alpha}-\boldsymbol{\beta}}, E_{\boldsymbol{\alpha}-\boldsymbol{\beta}}, E_{-\boldsymbol{\beta}}\}\,. \tag{4.26}$$

In this case $\boldsymbol{\beta}$ is a long root and

$$\boldsymbol{\alpha} = (2\boldsymbol{\alpha} - \boldsymbol{\beta}) + (\boldsymbol{\beta} - \boldsymbol{\alpha}) \qquad \Rightarrow \qquad \boldsymbol{\alpha}^\star = 2(2\boldsymbol{\alpha} - \boldsymbol{\beta})^\star + (\boldsymbol{\beta} - \boldsymbol{\alpha})^\star. \tag{4.27}$$

The condition that $0 < |y| < 1$ is equivalent to the requirement that

$$\mathrm{Re}(\boldsymbol{a}') \cdot (2\boldsymbol{\alpha} - \boldsymbol{\beta}) > 0 \qquad \text{and} \qquad \mathrm{Re}(\boldsymbol{a}') \cdot (\boldsymbol{\beta} - \boldsymbol{\alpha}) > 0, \tag{4.28}$$

and therefore implies that both $2\boldsymbol{\alpha}-\boldsymbol{\beta}$ and $\boldsymbol{\beta}-\boldsymbol{\alpha}$ are positive roots with respect to $\mathrm{Re}(\boldsymbol{a}')$, as required. The number of bosonic zero modes is 8 which is consistent with a decay into 3 stable monopoles.

(iii) $t = \frac{3}{2}$, which requires $\boldsymbol{\alpha}$ to be a short root. (This example only occurs in the case when the gauge group is G_2.) In this case the pair of multiplets consists of generators

$$\{E_{\boldsymbol{\beta}}, E_{\boldsymbol{\beta}-\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}-2\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}-3\boldsymbol{\alpha}}\} \qquad \text{and} \qquad \{E_{3\boldsymbol{\alpha}-\boldsymbol{\beta}}, E_{2\boldsymbol{\alpha}-\boldsymbol{\beta}}, E_{\boldsymbol{\alpha}-\boldsymbol{\beta}}, E_{-\boldsymbol{\beta}}\}\,. \tag{4.29}$$

In this case $\boldsymbol{\beta}$ is a long root and there are two possible decays

$$\begin{aligned} \boldsymbol{\alpha} &= (2\boldsymbol{\beta} - 3\boldsymbol{\alpha}) + 2(2\boldsymbol{\alpha} - \boldsymbol{\beta}) \qquad &\Rightarrow \qquad \boldsymbol{\alpha}^\star &= 3(2\boldsymbol{\beta} - 3\boldsymbol{\alpha})^\star + 2(2\boldsymbol{\alpha} - \boldsymbol{\beta})^\star \\ \boldsymbol{\alpha} &= (3\boldsymbol{\alpha} - \boldsymbol{\beta}) + (\boldsymbol{\beta} - 2\boldsymbol{\alpha}) \qquad &\Rightarrow \qquad \boldsymbol{\alpha}^\star &= 3(3\boldsymbol{\alpha} - \boldsymbol{\beta})^\star + (\boldsymbol{\beta} - 2\boldsymbol{\alpha})^\star. \end{aligned} \tag{4.30}$$

It is important in the above that all the terms in brackets are actually roots of the algebra. The first expression for $\boldsymbol{\alpha}^\star$ in (4.30) is relevant to the case when $0 < |y| < \frac{1}{2}$ which is equivalent to the requirement that

$$\mathrm{Re}(\boldsymbol{a}') \cdot (2\boldsymbol{\beta} - 3\boldsymbol{\alpha}) > 0 \qquad \text{and} \qquad \mathrm{Re}(\boldsymbol{a}') \cdot (2\boldsymbol{\alpha} - \boldsymbol{\beta}) > 0, \tag{4.31}$$

and therefore implies that both $2\boldsymbol{\beta} - 3\boldsymbol{\alpha}$ and $2\boldsymbol{\alpha} - \boldsymbol{\beta}$ are positive roots with respect to $\mathrm{Re}(\boldsymbol{a}')$, as required. The number of bosonic zero modes is 16 which is consistent with a decay into 5 stable monopoles.

The second expression in for $\boldsymbol{\alpha}^\star$ (4.30) is relevant to the case when $\frac{1}{2} < |y| < \frac{3}{2}$ which is equivalent to the requirement that

$$\mathrm{Re}(\boldsymbol{a}') \cdot (\boldsymbol{\beta} - 2\boldsymbol{\alpha}) > 0 \qquad \text{and} \qquad \mathrm{Re}(\boldsymbol{a}') \cdot (3\boldsymbol{\alpha} - \boldsymbol{\beta}) > 0, \tag{4.32}$$

and therefore implies that both $\boldsymbol{\beta} - 2\boldsymbol{\alpha}$ and $3\boldsymbol{\alpha} - \boldsymbol{\beta}$ are positive roots with respect to $\mathrm{Re}(\boldsymbol{a}')$, as required. The number of bosonic zero modes is 12 which is consistent with a decay into 4 stable monopoles.

Although the explicit expression for $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ can only be written down on a case-by-case basis, it is important that in all cases, as one can find by inspecting (4.23), (4.27) and (4.30), $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ can always be expressed as a linear combination of the two vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This fact will play a crucial role in our argument below. Another important consistency check of the picture is that the number of zero modes always matches the degrees-of-freedom implied by the number of decay products.

Now that we have described the zero modes of the real Higgs model, the next question concerns their fate when a complex Higgs theory is considered. Recall, that a zero mode is lifted if the eigenvalue of $\mathrm{Im}(\Phi) = \mathrm{Im}(\boldsymbol{a}') \cdot \boldsymbol{H}$ on the mode is non-zero. The eigenvalue associated to the $\boldsymbol{\beta}$ multiplet is

$$\mathrm{Im}(\boldsymbol{a}') \cdot \boldsymbol{\beta} = \frac{1}{|\boldsymbol{a} \cdot \boldsymbol{\alpha}|} \mathrm{Im}\left((\boldsymbol{a}^* \cdot \boldsymbol{\alpha})(\boldsymbol{\alpha} \cdot \boldsymbol{\beta}) \right), \tag{4.33}$$

and the conjugate multiplet has the opposite eigenvalue.

Notice that the $t = 1$ and $y = 0$ multiplet associated to the embedded $su(2)$ (2.13) is never lifted because $\mathrm{Im}(\boldsymbol{a}') \cdot \boldsymbol{\alpha} = 0$. This was to be expected since is corresponds to the freedom to shift the centre-of-mass and perform an overall U(1) charge rotation on the solution. The other multiplets are generally lifted and so the spherically symmetric monopole solution generically has no additional zero modes. However, on curves of co-dimension one in the moduli space of vacua, whenever $\mathrm{Im}(\boldsymbol{a}') \cdot \boldsymbol{\beta} = 0$, for some root $\boldsymbol{\beta}$ with $\boldsymbol{\beta} \cdot \boldsymbol{\alpha} \neq 0$, then additional zero modes appear that describe the freedom to deform the solution into $K + 1$ well separated constituent stable monopoles. On this special curve the $\boldsymbol{\alpha}$ monopole is at threshold for the decay:

$$|\boldsymbol{\alpha}^\star \cdot \boldsymbol{a}| = N\, |\boldsymbol{\gamma}^\star \cdot \boldsymbol{a}| + M\, |\boldsymbol{\delta}^\star \cdot \boldsymbol{a}|\,, \tag{4.34}$$

or

$$\frac{\boldsymbol{\gamma}\cdot\boldsymbol{a}'}{\boldsymbol{\delta}\cdot\boldsymbol{a}'}\equiv\frac{\boldsymbol{\gamma}\cdot\boldsymbol{a}}{\boldsymbol{\delta}\cdot\boldsymbol{a}}\in\mathbb{R}\geq 0. \tag{4.35}$$

This follows from the fact that since $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ can be expanded in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and $\mathrm{Im}(\boldsymbol{a}')\cdot\boldsymbol{\alpha}=\mathrm{Im}(\boldsymbol{a}')\cdot\boldsymbol{\beta}=0$ necessarily implies $\mathrm{Im}(\boldsymbol{a}')\cdot\boldsymbol{\gamma}=\mathrm{Im}(\boldsymbol{a}')\cdot\boldsymbol{\delta}=0$. Notice that (4.35) coincides with the definition of $C_{\boldsymbol{\gamma},\boldsymbol{\delta}}$ in (3.4).

There is one remaining point to clear up. The decay products $\boldsymbol{\alpha}^\star=N\boldsymbol{\gamma}^\star+M\boldsymbol{\delta}^\star$ that are associated to the two roots $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$, are required to be positive roots with respect to the simple roots $\boldsymbol{\alpha}_i'$ defined with respect to $\mathrm{Re}(\boldsymbol{a}')$. However in section 3 we determined that the decay can only proceed if the decay products where associated to positive roots with respect to the simple roots $\boldsymbol{\alpha}_i$ defined with respect to $\boldsymbol{a}^1\equiv\mathrm{Re}(\boldsymbol{a})$. This is consistent because precisely on the CMS the two roots $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are positive roots with respect to *both* definitions of simple root. To see this consider

$$\mathrm{Re}(\boldsymbol{a}')\cdot\boldsymbol{\gamma}=\frac{1}{|\boldsymbol{a}\cdot\boldsymbol{\alpha}|}\mathrm{Re}\left((\boldsymbol{a}\cdot\boldsymbol{\gamma})(\boldsymbol{a}^*\cdot\boldsymbol{\alpha})\right)=\frac{\boldsymbol{\alpha}^2}{|\boldsymbol{a}\cdot\boldsymbol{\alpha}|}\mathrm{Re}\left(N\frac{|\boldsymbol{a}\cdot\boldsymbol{\gamma}|^2}{\boldsymbol{\gamma}^2}+M\frac{(\boldsymbol{a}\cdot\boldsymbol{\gamma})(\boldsymbol{a}^*\cdot\boldsymbol{\delta})}{\boldsymbol{\delta}^2}\right), \tag{4.36}$$

But on the CMS curve $(\boldsymbol{a}\cdot\boldsymbol{\gamma})(\boldsymbol{a}^*\cdot\boldsymbol{\delta})$ is a positive real number, therefore we deduce that $\mathrm{Re}(\boldsymbol{a}')\cdot\boldsymbol{\gamma}$ is a positive number and consequently on the $C_{\boldsymbol{\gamma},\boldsymbol{\delta}}$ the root $\boldsymbol{\gamma}$ is also a positive root with respect to the simple roots $\boldsymbol{\alpha}_i'$. An identical argument follows for $\boldsymbol{\delta}$.

5. The dynamics of monopole decay

In section 3, we considered the CMS of the classical moduli space of vacua W where it was energetically possible for dyons to decay. In the present section we now pose the dynamical question as to whether dyons do actually decay on these CMS. Although some pieces of the argument have not been fully proven for all possible gauge groups the resulting picture is rather convincing: in an $N=2$ pure gauge theory at weak coupling dyons always decay on their CMS, whereas in the related $N=4$ theory they do not decay. This is one manifestation of the relative simplicity of $N=4$ theories over $N=2$ theories.

The analysis in the last section has all been at the level of zero modes around the spherically symmetric monopoles solutions. From this, we have concluded that these solutions, obtained by embeddings of the SU(2) monopole, are generically stable in W. The only zero modes which are generically present account for the centre-of-mass and U(1) charge angle of the solution. On the submanifold (4.35), of co-dimension one, $4(N+M-1)$ modes are destabilized and become zero modes. These modes reflect that fact that the monopole moduli space includes deformations away from the spherically symmetric solution leading to an enlargement of the moduli space by a factor $\mathcal{M}_0$ of dimension $4(N+M-1)$. In the case when $N=M=1$ corresponding to zero modes with $t=\frac{1}{2}$, $\mathcal{M}_0$ is known explicitly to be a Euclidean Taub-NUT manifold [12,14,15]. However for $t=1$ or $t=\frac{3}{2}$, the dimension

of $\mathcal{M}_0$ is 8 and 12 or 16, respectively, and the manifold is only known in asymptotic regions [12]. In all cases, however, $\mathcal{M}_0$ is a hyper-Kähler manifold [11].

We now make the hypothesis that the spectra of $N = 4$ theories manifest exact GNO duality for all gauge groups. The idea is that at all points in the moduli space of vacua in an $N = 4$ supersymmetric gauge theory, the spectrum of monopoles must match the spectrum of gauge bosons in the dual theory, defined as an $N = 4$ supersymmetric gauge theory but whose gauge group has a Lie algebra $g^\star$ being the "dual" of the original Lie algebra g. The dual algebra is defined as the algebra whose simple roots are $\boldsymbol{\alpha}_i^\star = \boldsymbol{\alpha}_i/\boldsymbol{\alpha}_i^2$, up to some overall normalization. Hence, all the simply-laced theories are self-dual, whereas the algebra/dual algebra relationships of the non-simply-laced theories are

$$\begin{aligned} sp(n) &\leftrightarrow so(2n+1) \\ f_4 &\leftrightarrow f_4' \\ g_2 &\leftrightarrow g_2', \end{aligned} \tag{5.1}$$

where the prime indicates a re-labelling of the roots. Since in the dual theory there is a gauge boson associated to every root $\boldsymbol{\alpha}^\star$ of $g^\star$, with electric charge $Q_E^I = \boldsymbol{a}^I \cdot \boldsymbol{\alpha}^\star$, for GNO duality to be reflected in the spectrum there must be a monopole of magnetic charge $Q_M^I = \boldsymbol{a}^I \cdot \boldsymbol{\alpha}^\star$ at all points in the moduli space of vacua W. In other words, in the $N = 4$ theory monopoles do not decay on their CMS and consequently they must exist on these submanifolds as bound-states at threshold. Such states can indeed exists within the semi-classical approximation if there exists a square-integrable harmonic form on each $\mathcal{M}_0$. In the cases where $\mathcal{M}_0$ is known (for $N = M = 1$ i.e. $t = \frac{1}{2}$) the appropriate harmonic form has been found [12,14,15,18], thus providing strong evidence for GNO duality.

If GNO duality is exact in the associated $N = 4$ theory, i.e. the bound-states at threshold exist, it implies that the corresponding dyons in an $N = 2$ theory actually decay on their CMS. To appreciate this, we have to consider the difference between the quantum mechanics that arises on $\mathcal{M}_0$ in the context of the semi-classical approximation of monopoles in an $N = 2$ and $N = 4$ supersymmetric gauge theory, respectively. In both cases the space-time supersymmetry is manifested by supersymmetries in the quantum mechanics on $\mathcal{M}_0$. For $N = 4$ there are "$4 \times 1/2$" supersymmetries [26] whilst for $N = 2$ there are "$2 \times 1/2$" supersymmetries [11]. In both cases states can be associated to differential forms on $\mathcal{M}_0$, however for $N = 4$ the states correspond to any form whilst in $N = 2$ they correspond only to holomorphic forms.[3] Bound-states correspond to normalizable ground-states of the quantum mechanics which in turn correspond to square-integrable harmonic forms on $\mathcal{M}_0$. As we have argued, exact duality implies that there is a unique square-integrable harmonic form on $\mathcal{M}_0$ and since a holomorphic harmonic form would inevitably have an anti-holomorphic partner, we conclude that there cannot be a square-integrable holomorphic harmonic form on $\mathcal{M}_0$. So there are no bound-states of monopoles in an $N = 2$ theory (with no matter) and the dyons in these theories must decay on their CMS

[3] Recall that $\mathcal{M}_0$ is a (hyper) Kähler manifold.

at weak coupling. As we have mentioned, the statement that monopoles in the $N = 2$ theory actually decay can really only be proved for decays with $N = M = 1$, i.e. when there are only two dyons in the final state. Nevertheless, this is enough to prove that monopoles decay in *all* the simply-laced theories, where, generically, all the decays have two dyons in the final state.

Exactly the same line of argument should prove that there are no stable dyons in an $N = 2$ theory whose magnetic charge $Q^I_M = \boldsymbol{a}^I \cdot \boldsymbol{\beta}^\star$ are given by vectors $\boldsymbol{\beta}$ which are not roots of g. Such states do exist in the associated $N = 4$ theory, with $\boldsymbol{\beta} = n\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is a root of g and $n \in \mathbb{Z} > 1$ [14,27]. Given the uniqueness of these bound-states in the $N = 4$ theory, which follows from the analysis in the SU(2) theory [28,29], they will not occur in the $N = 2$ theory.

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